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ScienceDirect
Appl. Comput. Harmon. Anal. 21 (2006) 413–420
Applied and Computational Harmonic Analysis
www.elsevier.com/locate/acha

Letter to the Editor

Approximation of bandlimited functions

Yoel Shkolnisky [a,1], Mark Tygert [b,2], Vladimir Rokhlin [c,3]

[a] *Department of Mathematics, Yale University, 51 Prospect St., A.K. Watson Room 113, New Haven, CT 06511, USA*
[b] *Program in Applied Mathematics, Yale University, 51 Prospect St., A.K. Watson Room 110, New Haven, CT 06511, USA*
[c] *Departments of Computer Science, Mathematics, and Physics, Yale University, 51 Prospect St., A.K. Watson Room 108, New Haven, CT 06511, USA*

Available online 9 June 2006

Communicated by Leslie F. Greengard on 11 February 2006

Abstract

Many signals encountered in science and engineering are approximated well by bandlimited functions. We provide suitable error bounds for the approximation of bandlimited functions by linear combinations of certain special functions—the prolate spheroidal wave functions of order 0. The coefficients in the approximating linear combinations are given explicitly via appropriate quadrature formulae.

Keywords: Bandlimited; Approximation; Interpolation; Prolate spheroidal wave functions

1. Introduction

Bandlimited functions arise frequently in many scientific and engineering contexts, particularly those involving signal processing. A *bandlimited* function f is a function such that

$$f(x) = \int_{-1}^{1} e^{ic\xi x}\, \mathrm{d}\mu(\xi), \tag{1}$$

for some complex-valued measure μ with $\int_{-1}^{1} \mathrm{d}|\mu(\xi)| < \infty$, where c is a positive real number known as the *bandlimit* of f.

E-mail addresses: yoel.shkolnisky@yale.edu (Y. Shkolnisky), mark.tygert@yale.edu (M. Tygert).
[1] Supported in part by a grant from the Ministry of Science, Israel.
[2] Supported in part by the U.S. DoD, under a 2001 NDSEG fellowship.
[3] Supported in part by the U.S. DoD, under AFOSR Grant #F49620-03-C-0041.

1063-5203/$ – see front matter
doi:10.1016/j.acha.2006.05.001

In practical circumstances, it is often desirable to approximate a function f of bandlimit c as a linear combination of standardized basis functions, say $\psi_0^c, \psi_1^c, \ldots, \psi_{K-1}^c, \psi_K^c$, via the formula

$$f(x) \approx \sum_{k=0}^{K} f_k^c \psi_k^c(x), \tag{2}$$

where $f_0^c, f_1^c, \ldots, f_{K-1}^c, f_K^c$ are appropriately defined complex numbers.

The present note constructs approximations of the form (2) when $\psi_0^c, \psi_1^c, \psi_2^c, \ldots$ are the prolate spheroidal wave functions of bandlimit c and order 0. The coefficients $f_0^c, f_1^c, \ldots, f_{K-1}^c, f_K^c$ in (2) are obtained directly from the values $f(x_1), f(x_2), \ldots, f(x_{N-1}), f(x_N)$ via the formula

$$f_k^c = \sum_{n=1}^{N} w_n \psi_k^c(x_n) f(x_n), \tag{3}$$

where $x_0, x_1, \ldots, x_{N-1}, x_N$ are quadrature nodes in $[-1, 1]$ and $w_1, w_2, \ldots, w_{N-1}, w_N$ are quadrature weights such that

$$\int_{-1}^{1} g(x)\,\mathrm{d}x \approx \sum_{n=1}^{N} w_n g(x_n) \tag{4}$$

for any function g with bandlimit $2c$.

Moreover, the number N of quadrature nodes $x_1, x_2, \ldots, x_{N-1}, x_N$ in (3) and (4) is approximately equal to the number $K+1$ of basis functions $\psi_0^c, \psi_1^c, \ldots, \psi_{K-1}^c, \psi_K^c$ in (2), as one would expect with an optimal sampling scheme.

For discussion of similar numerical schemes optimized for use with bandlimited functions, see [1,2,5]. The error bounds found in [5] break down in certain circumstances; the purpose of the present note is to provide suitable error bounds for the type of approximation first introduced in [5].

The present note has the following structure: Section 2 summarizes several well-known facts about prolate spheroidal wave functions and sets the notation used throughout the present note. Section 3 constructs a scheme for approximating bandlimited functions.

2. Preliminaries

This section summarizes certain well-known properties of prolate spheroidal wave functions. All of these can be found in [4,5]. The notation introduced in the present section will be used without further notice throughout the remainder of the present note.

For any positive real number c, the function $F(x, \xi) = e^{ic\xi x}$ is a self-adjoint Hilbert–Schmidt kernel on $[-1, 1] \times [-1, 1]$; as such, it has a sequence of eigenfunctions $\psi_0^c, \psi_1^c, \psi_2^c, \ldots$ which are orthonormal and complete in $\mathcal{L}^2([-1, 1])$, with corresponding eigenvalues $\lambda_0^c, \lambda_1^c, \lambda_2^c, \ldots,$ so that

$$\int_{-1}^{1} e^{ic\xi x} \psi_k^c(\xi)\,\mathrm{d}\xi = \lambda_k^c \psi_k^c(x) \tag{5}$$

for any $x \in [-1, 1]$ and nonnegative integer k. All the eigenvalues $\lambda_0^c, \lambda_1^c, \lambda_2^c, \ldots$ have multiplicity 1; in the present note the eigenvalues are ordered so that $|\lambda_0^c| > |\lambda_1^c| > |\lambda_2^c| > \cdots$. The functions $\psi_0^c, \psi_1^c, \psi_2^c, \ldots$ are real-valued and are known as the prolate spheroidal wave functions of order 0 (PSWFs).

As k increases within the integers in a band centered about $2c/\pi$ of width proportional to $\ln c$, $|\lambda_k^c|$ decays from very nearly $\sqrt{2\pi/c}$ to very nearly 0; $|\lambda_k^c| \approx \sqrt{2\pi/c}$ when k is a nonnegative integer below this band and $|\lambda_k^c| \approx 0$ when k is an integer above the band. The following theorem, proven (in a slightly different form) in [3], states this more precisely.

Theorem 2.1. *Suppose that ε is a real number such that* $0 < \varepsilon < 1$.

Then,

$$\lim_{c\to\infty}\left|\frac{2}{\pi}c+\frac{1}{\pi^2}\left(\ln\frac{1-\varepsilon}{\varepsilon}\right)\ln c-K\right|\Big/\ln c=0, \tag{6}$$

where $K=K(c)$ is the greatest integer such that

$$\left|\lambda_K^c\right|\geqslant\sqrt{\frac{2\pi}{c}}\,\varepsilon. \tag{7}$$

The following lemma, providing an expression for λ_k^c in terms of $\psi_k^b(1)$ with $0<b<c$, is stated as formula 57 in [4].

Lemma 2.2. *Suppose that c is a positive real number and k is a nonnegative integer.*
Then,

$$\lambda_k^c=\frac{i^k\sqrt{\pi}c^k(k!)^2}{(2k)!\Gamma(k+\frac{3}{2})}\exp\int_0^c\left(\frac{2(\psi_k^b(1))^2-1}{2b}-\frac{k}{b}\right)\mathrm{d}b, \tag{8}$$

where Γ is the gamma (factorial) function.

The following lemma, providing a bound for $\psi_k^c(1)$, is stated as formula 66 in [4]; details of its proof will be reported at a later date.

Lemma 2.3. *Suppose that c is a positive real number and k is a nonnegative integer.*
Then,

$$\left|\psi_k^c(1)\right|<\sqrt{k+\frac{1}{2}}. \tag{9}$$

For any positive real number c and nonnegative integer k, M_k^c will denote the maximum value

$$M_k^c=\max_{0\leqslant j\leqslant k}\max_{-1\leqslant x\leqslant 1}\left|\psi_j^c(x)\right|. \tag{10}$$

Remark 2.4. Analysis similar to that mentioned in [4, Section 5] yields the bound

$$M_k^c\leqslant 2\sqrt{k} \tag{11}$$

for any positive real number c and positive integer k. A detailed proof of (11) will be reported at a later date.

For any positive integer N and positive real numbers c and ε, *generalized Gaussian quadrature nodes* $x_1, x_2, \ldots, x_{N-1}, x_N$ and *associated quadrature weights* $w_1, w_2, \ldots, w_{N-1}, w_N$, *which integrate on* $[-1,1]$ *functions with bandlimit* c, *to precision* ε, are real numbers x_n and w_n such that $x_n\in[-1,1]$ for $n=1,2,\ldots,N-1,N$, and

$$\left|\int_{-1}^{1}f(x)\,\mathrm{d}x-\sum_{n=1}^{N}w_n f(x_n)\right|<\varepsilon\int_{-1}^{1}\mathrm{d}|\mu|(\xi) \tag{12}$$

for any $x\in[-1,1]$, complex-valued measure μ, and function f on $[-1,1]$ of the form (1).

Remark 2.5. As shown in [5], there exist two different methods for generating generalized Gaussian quadrature nodes and weights which integrate bandlimited functions, the first using the zeros of PSWFs, and the second using quadratures that are exact for sets of PSWFs. Justification that the first method works can be found in [1], while a proof that the second method works can be found in [2]. Copious numerical examples and applications of quadratures generated by both methods can be found in [5].

3. Approximation of bandlimited functions with linear combinations of PSWFs

This section approximates a bandlimited function f with a linear combination of PSWFs, using the values of f at certain nodes. Section 3.1 provides a bound for the error arising from using only a finite number of PSWFs in the approximating linear combination. Sections 3.2 and 3.3 provide a bound for the error arising from calculating the coefficients in the approximating linear combination from the values of f at certain quadrature nodes. Section 3.4 combines the results of Sections 3.1–3.3 to obtain the principal result of the present note.

3.1. Truncation of infinite series of PSWFs

For any positive real number c, since (as mentioned in Section 2) the PSWFs are orthonormal and complete in $\mathcal{L}^2([-1,1])$, any function f with bandlimit c has an $\mathcal{L}^2$-convergent expansion

$$f(x) = \sum_{k=0}^{\infty} p_k^c \psi_k^c(x), \tag{13}$$

where p_k^c is the projection of f onto ψ_k^c, defined by

$$p_k^c = \int_{-1}^{1} \psi_k^c(x) f(x)\,\mathrm{d}x. \tag{14}$$

This section provides a bound for $|p_k^c|$ and thence for the error arising from truncating the infinite sum in (13).

The following lemma provides a bound for the magnitude of the projection of a bandlimited function onto a PSWF.

Lemma 3.1. *Suppose that c is a positive real number, k is a nonnegative integer, and f is a function on* $[-1,1]$ *of the form* (1), *for some complex-valued measure μ.*

Then,

$$\left|p_k^c\right| \leqslant \left|\lambda_k^c\right| M_k^c \int_{-1}^{1} \mathrm{d}|\mu|(\xi), \tag{15}$$

where p_k^c is the projection of f onto ψ_k^c, defined in (14).

Proof. Substituting (1) into (14), exchanging the order of integration, and using (5) yields (15). □

The following lemma provides an extremely rough bound on a certain series which relates to the tail of the series in (13).

Lemma 3.2. *Suppose that c is a positive real number and K is a positive integer such that*

$$K \geqslant 2c. \tag{16}$$

Then,

$$\sum_{k=K+1}^{\infty} \left|\lambda_k^c\right| \left(M_k^c\right)^2 \leqslant \frac{8(K+2)}{2^K}. \tag{17}$$

Proof. Combining (8) and (9) yields that

$$\left|\lambda_k^c\right| \leqslant \frac{\sqrt{\pi}\, c^k (k!)^2}{(2k)!\,\Gamma(k+\frac{3}{2})} \tag{18}$$

for any nonnegative integer k. It follows from (18) that

$$\left|\lambda_k^c\right| \leqslant \frac{2}{2^k} \tag{19}$$

for any integer k with $k \geqslant 2c$. Combining (19), (16), and (11) yields (17). □

Remark 3.3. The requirement in (16) is not tight; (17) serves only to show that the series in (13) satisfies the Cauchy criterion for convergence.

The following lemma states that (13) is valid uniformly on $[-1,1]$.

Lemma 3.4. *Suppose that c is a positive real number and f is a function on* $[-1,1]$ *of the form* (1)*, for some complex-valued measure μ.*

Then, the series

$$\sum_{k=0}^{\infty} p_k^c \psi_k^c(x) \tag{20}$$

converges uniformly to f on $[-1,1]$*, where p_k^c is the projection of f onto ψ_k^c, defined in* (14).

Proof. The series (20) converges uniformly since, due to (15) and (17), it satisfies the Cauchy criterion for uniform convergence, i.e., given any positive real number ε, there exists an integer K such that

$$\left| \sum_{k=N_0}^{N_1} p_k^c \psi_k^c(x) \right| \leqslant \varepsilon \tag{21}$$

for any $x \in [-1,1]$ and integers N_0 and N_1 with $N_1 \geqslant N_0 \geqslant K$.

Combining the fact that (13) is valid in the $\mathcal{L}^2$-sense with the fact that (20) converges uniformly on $[-1,1]$ yields that (13) is valid uniformly on $[-1,1]$. $\square$

Combining (13), (15), and Lemma 3.4 yields the following lemma, providing a bound for the error involved in approximating a bandlimited function with a truncated expansion of the function into PSWFs.

Lemma 3.5. *Suppose that c is a positive real number, K is a nonnegative integer, and f is a function on* $[-1,1]$ *of the form* (1)*, for some complex-valued measure μ.*

Then,

$$\left| f(x) - \sum_{k=0}^{K} p_k^c \psi_k^c(x) \right| \leqslant \left(\sum_{k=K+1}^{\infty} \left|\lambda_k^c\right| \left(M_k^c\right)^2 \right) \int_{-1}^{1} \mathrm{d}|\mu|(\xi) \tag{22}$$

for any $x \in [-1,1]$, where p_k^c is the projection of f onto ψ_k^c, defined in (14).

Table 1 lists the errors due to truncating the expansion (13) after K terms. That is, Table 1 lists numerical values of the series

$$\sum_{k=K+1}^{\infty} \left|\lambda_k^c\right| \left(M_k^c\right)^2 \tag{23}$$

from the right-hand side of (22), for the specified values of c and K.

3.2. Product of a PSWF and a bandlimited function

Applying quadratures to the integrand $\psi_k^c(x) f(x)$ in (14) provides an approximation to the projection p_k^c of f onto ψ_k^c. Clearly, the quadratures employed must be appropriate for this particular integrand. This section shows that the integrand $\psi_k^c(x) f(x)$ is bandlimited, with bandlimit $2c$.

The following lemma, a consequence of (5), states that a PSWF is bandlimited.

Lemma 3.6. *Suppose that c is a positive real number and k is a nonnegative integer.*

Table 1
Numerical values of the truncation error factor (23) from (22)

c	Truncation error	$2KM_K^c\lvert\lambda_K^c\rvert$	K	$2c/\pi$
1.0	0.421321E–15	0.718404E–13	13	0.6
2.0	0.158369E–15	0.184668E–13	16	1.3
3.0	0.216360E–15	0.199890E–13	18	1.9
10.0	0.179820E–15	0.936746E–14	28	6.4
20.0	0.381179E–15	0.150861E–13	38	12.7
30.0	0.348955E–15	0.122880E–13	47	19.1
100.0	0.512130E–15	0.147296E–13	100	63.7
150.0	0.523688E–15	0.148763E–13	135	95.5
200.0	0.628518E–15	0.179342E–13	169	127.3
250.0	0.430685E–15	0.125057E–13	203	159.2
300.0	0.632490E–15	0.185801E–13	236	191.0
1000.0	0.633588E–15	0.226427E–13	692	636.6
1500.0	0.595142E–15	0.233808E–13	1014	954.9
2000.0	0.554380E–15	0.234477E–13	1335	1273.2
2500.0	0.716780E–15	0.321516E–13	1655	1591.5
3000.0	0.703437E–15	0.332122E–13	1975	1909.9

Then,

$$\psi_k^c(x) = \int_{-1}^{1} e^{ic\xi x}\,\mathrm{d}\mu_k(\xi) \tag{24}$$

for any $x \in [-1, 1]$, *where the complex-valued measure* μ_k *is defined by*

$$\mathrm{d}\mu_k(\xi) = \mathrm{d}_k(\xi)\,\mathrm{d}\xi, \tag{25}$$

with the function d_k *defined on* $[-1, 1]$ *by*

$$\mathrm{d}_k(\xi) = \frac{\psi_k^c(\xi)}{\lambda_k^c}. \tag{26}$$

Furthermore,

$$\int_{-1}^{1} \mathrm{d}|\mu_k|(\xi) \leqslant \frac{2M_k^c}{|\lambda_k^c|}. \tag{27}$$

The representation (24) yields the following lemma; this lemma states that the product of a bandlimited function and a PSWF is bandlimited, with twice the bandlimit of the bandlimited function and the PSWF factors in the product.

Lemma 3.7. *Suppose that c is a positive real number, k is a nonnegative integer, and f is a function on* $[-1, 1]$ *of the form* (1), *for some complex-valued measure* μ.

Then,

$$\psi_k^c(x) f(x) = \int_{-1}^{1} e^{i(2c)\xi x}\,\mathrm{d}\pi_k(\xi) \tag{28}$$

for any $x \in [-1, 1]$, *where the complex-valued measure* π_k *is defined by*

$$\mathrm{d}\pi_k(\xi) = 2\left(\int_{-1}^{1} \mathrm{d}_k(2\xi - \eta)\,\mathrm{d}\mu(\eta)\right)\mathrm{d}\xi, \tag{29}$$

with the function d_k *defined on* $(-\infty, \infty)$ *by*

$$\mathrm{d}_k(\xi) = \frac{\psi_k^c(\xi)}{\lambda_k^c} \tag{30}$$

when $-1 \leqslant \xi \leqslant 1$, *and*

$$\mathrm{d}_k(\xi) = 0 \tag{31}$$

otherwise (*when* $\xi < -1$ *or* $\xi > 1$).

Furthermore,

$$\int_{-1}^{1} \mathrm{d}|\pi_k|(\xi) \leqslant \frac{2M_k^c}{|\lambda_k^c|} \int_{-1}^{1} \mathrm{d}|\mu|(\xi). \tag{32}$$

Remark 3.8. As mentioned earlier, applying quadratures to the integrand $\psi_k^c(x) f(x)$ in (14) yields an approximation to p_k^c. Provided that (22) is utilized, an approximation to p_k^c is needed only when $k \leqslant K$, where K is the index of the eigenvalue λ_K^c for which $|\lambda_K^c| \sim \varepsilon$, and ε is the precision of computations. Therefore, (27) and (32) are needed only when $k \leqslant K$. Due to (11), for any positive real number ε and positive integer k, if the eigenvalue λ_k^c has the bound

$$\left|\lambda_k^c\right| \geqslant \varepsilon, \tag{33}$$

then the factor in the right-hand side of (32) has the rather poor bound

$$\frac{2M_k^c}{|\lambda_k^c|} \leqslant \frac{4\sqrt{k}}{\varepsilon}. \tag{34}$$

Section 3.3 compensates for this rather poor bound by using very accurate quadratures, involving sufficiently many quadrature nodes.

3.3. Approximation via quadratures to the projection of a bandlimited function onto a PSWF

This section applies quadratures to the integrand $\psi_k^c(x) f(x)$ in (14) in order to approximate the projection p_k^c of f onto ψ_k^c.

The bound (32) yields the following lemma, which applies generalized Gaussian quadratures to the integrand $\psi_k^c(x) f(x)$ in (14), thus obtaining a good approximation $\tilde{p}_k^c$ to the projection p_k^c of f onto ψ_k^c. The quadratures employed integrate on $[-1, 1]$ functions with bandlimit $2c$, to precision $|\lambda_K^c|^2$.

Lemma 3.9. *Suppose that* k *is a nonnegative integer,* K *and* N *are positive integers,* c *is a positive real number, and* $x_1, x_2, \ldots, x_{N-1}, x_N$ *are generalized Gaussian quadrature nodes and* $w_1, w_2, \ldots, w_{N-1}, w_N$ *are associated quadrature weights, which integrate on* $[-1, 1]$ *functions with bandlimit* $2c$, *to precision* $|\lambda_K^c|^2$. *Suppose in addition that* $k \leqslant K$ *and* f *is a function on* $[-1, 1]$ *of the form* (1), *for some complex-valued measure* μ.

Then,

$$\left|p_k^c - \tilde{p}_k^c\right| \leqslant 2M_K^c \left|\lambda_K^c\right| \int_{-1}^{1} \mathrm{d}|\mu|(\xi), \tag{35}$$

where p_k^c *is the projection of* f *onto* ψ_k^c, *defined in* (14), *and* $\tilde{p}_k^c$ *is the approximation to* p_k^c *obtained by applying the quadratures to the integrand* $\psi_k^c(x) f(x)$; $\tilde{p}_k^c$ *is defined by*

$$\tilde{p}_k^c = \sum_{n=1}^{N} w_n \psi_k^c(x_n) f(x_n). \tag{36}$$

Remark 3.10. The bound (11) yields that, for any positive real number ε, if the eigenvalue λ_K^c has the bound

$$\left|\lambda_K^c\right| \leqslant \varepsilon, \tag{37}$$

then the factor in the right-hand side of (35) has the bound

$$2M_K^c\left|\lambda_K^c\right| \leqslant 4\sqrt{K}\varepsilon. \tag{38}$$

3.4. Approximation scheme for bandlimited functions

Combining (22) and (35) yields the following theorem; this theorem provides a bound for the error arising from approximating a bandlimited function with a linear combination of PSWFs, with the coefficients in the linear combination determined from values of the bandlimited function via (36).

Theorem 3.11. *Suppose that K and N are positive integers, c is a positive real number, $x_1, x_2, \ldots, x_{N-1}, x_N$ are generalized Gaussian quadrature nodes, $w_1, w_2, \ldots, w_{N-1}, w_N$ are associated quadrature weights, which integrate on $[-1,1]$ functions with bandlimit $2c$, to precision $|\lambda_K^c|^2$, and f is a function on $[-1,1]$ of the form (1), for some complex-valued measure μ.*

Then,

$$\left|f(x)-\sum_{k=0}^{K}\tilde{p}_k^c\psi_k^c(x)\right| \leqslant \left(2KM_K^c\left|\lambda_K^c\right| + \sum_{k=K+1}^{\infty}\left|\lambda_k^c\right|\left(M_k^c\right)^2\right)\int_{-1}^{1}\mathrm{d}|\mu|(\xi) \tag{39}$$

for any $x \in [-1,1]$, where $\tilde{p}_k^c$ is the approximation to the projection of f onto ψ_k^c, defined in (36).

Remark 3.12. See Remark 3.10 and Table 1 for bounds on the factors $2KM_K^c|\lambda_K^c|$ and $\sum_{k=K+1}^{\infty}|\lambda_k^c|(M_k^c)^2$ in the right-hand side of (39).

Acknowledgments

The authors thank R.R. Coifman and M. Maggioni for helpful discussions.

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